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SOLUTION OF A CONTACT PROBLEM FOR A PLATE WITH A DEFORMABLE INSERT

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Contact problems with friction are solved for a rectangular plate with a circular hole into which a ring plate (insert) is placed with a small clearance. Two versions of contact boundary conditions are formulated. According to the proposed approximate formulation of the problem, the boundary conditions in both versions are satisfied not at the actual contact points but at specified pairs of points. Therefore, it is sufficient to determine attachment, slip, contact, and contact-free regions on just one of the contacting contours. The finite-element method and the Boussinesq principle are used to solve the problem. One of the versions of boundary conditions, compared to the other, gives smaller values for the strain energies of the plate and insert, the stress-concentration coefficient, and the lengths of attachment and contact regions.

1. Basic Equations. The equations of equilibrium, the strain–displacement relations, and Hooke's law are written in the form [1]

$$
\begin{gathered}
\sigma_{11,1} + \sigma_{12,2} = 0, \qquad \sigma_{12,1} + \sigma_{22,2} = 0, \qquad e_{11} = u_{1,1} = E^{-1}(\sigma_{11} - \nu\sigma_{22}), \\
e_{22} = u_{2,2} = E^{-1}(\sigma_{22} - \nu\sigma_{11}), \qquad e_{12} = 0{,}5(u_{1,2} + u_{2,1}) = (1+\nu)E^{-1}\sigma_{12}.
\end{gathered}
\tag{1.1}
$$

Here E is the Young's modulus, ν is the Poisson's ratio, u_i are the displacements, e_{ij} are the strains, and σ_{ij} are the plane stresses in the Cartesian coordinates x_i (i, $j = 1$ and 2); subscripts 1 and 2 after a comma denote partial differentiation with respect to x_1 and x_2, respectively. The strain energy has the form

$$
\Phi_E = \int\limits_{\Omega} \frac{E}{2(1-\nu^2)}\,[e_{11}^2 + 2\nu e_{11}e_{22} + e_{22}^2 + 2(1-\nu)e_{12}^2]\,dx_1\,dx_2.
$$

It is assumed that the thickness of the plates is constant and, without loss in generality, equal to unity. Integration is performed over the region Ω occupied by the plate. For the rectangular plate, we have $E = E_1$ and $\nu = \nu_1$ and for the ring plate, $E = E_2$ and $\nu = \nu_2$.

2. Boundary Conditions Outside the Contact Region. We consider a rectangular plate of width $2H$ and length $L = L_1 + L_2$ with a circular hole of radius R, which will be referred to as a "plate," and a ring plate (insert) with outside and inside radii $R_1 = R - c$ and R_2, respectively, whose center is at the point with the Cartesian coordinates $(-c, 0)$, $c = \varepsilon R$, where ε is a small dimensionless clearance parameter ($\varepsilon > 0$). In view of symmetry, the solution is sought only for the upper halves of the plate and the insert [whose undeformed states with zero clearance ($\varepsilon = 0$) are shown in Fig. 1a] subject to the boundary conditions

$$
\begin{gathered}
\sigma_{11} = \sigma_{12} = 0 \quad \text{at} \quad x_1 = -L_1, \quad 0 \leqslant x_2 \leqslant H, \\
u_2 = 0, \quad \sigma_{12} = 0 \quad \text{at} \quad x_2 = \begin{cases} H, & -L_1 \leqslant x_1 \leqslant L_2, \\ 0, & -L_1 \leqslant x_1 \leqslant -R, \quad R \leqslant x_1 \leqslant L_2, \end{cases} \\
u_1 = w, \qquad u_2 = 0 \quad \text{at} \quad x_1 = L_2, \quad 0 \leqslant x_2 \leqslant H;
\end{gathered}
\tag{2.1}
$$

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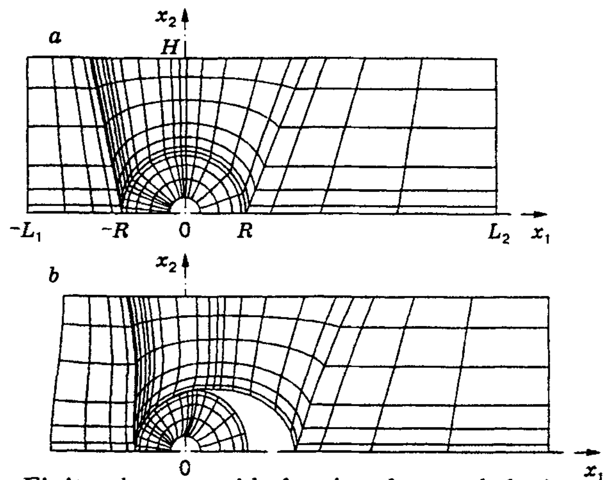

Fig. 1. Finite-element grids for the plate and the insert in the initial (a) and deformed (b) states for the contact problem without a clearance ($\varepsilon = 0$, $\mu = 0.3$, $E_1 = E_2$, $w = 1$, $l = 0.4807$, and $b = 0.1485$).

for the insert,

$$\begin{gathered} u_1 = u_2 = 0 \quad \text{at} \quad \rho = R_2, \quad 0 \leqslant \theta \leqslant \pi, \\ u_2 = 0, \quad \sigma_{12} = 0 \quad \text{for} \quad R_2 \leqslant \rho \leqslant R_1, \quad \theta = 0 \text{ and } \theta = \pi. \end{gathered} \tag{2.2}$$

The following two polar coordinate systems are used: (r, φ) ($x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$) and (ρ, θ) ($x_1 = \rho\cos\theta - c$ and $x_2 = \rho\sin\theta$).

The tensile force P applied on the right side of the plate and the work Φ done by this force are given by the formulas

$$P = \int_0^H \sigma_{11}\, dx_2, \qquad \Phi = \int_0^\tau P\dot{w}\, d\tau \quad \text{for} \quad x_1 = L_2.$$

The dot denotes differentiation with respect to the loading parameter, which is called time τ. Below, we also formulate boundary conditions on the contour of the insert Γ_c ($\rho = R_1$ and $0 \leqslant \theta \leqslant \pi$) and on the hole contour Γ_p ($r = R$ and $0 \leqslant \varphi \leqslant \pi$).

3. Work of Frictional Forces in Displacement Variations. According to the virtual displacement principle, for any displacement variations δu_i satisfying the boundary conditions for the displacements (2.1) and (2.2) and for any associated strain variations δe_{ij} the stress works in the plate and the insert is equal to the work of the forces acting on them:

$$\delta\Phi_{E1} = \delta\Phi - \delta\Phi_{\Gamma 1}, \qquad \delta\Phi_{E2} = \delta\Phi_{\Gamma 2}. \tag{3.1}$$

Here

$$\delta\Phi_{E1} = \int_{\Omega_1} \sigma_{ij}\delta e_{ij}\, dx_1\, dx_2, \qquad \delta\Phi_{E2} = \int_{\Omega_2} \sigma_{ij}\delta e_{ij}\, dx_1\, dx_2,$$

$$\delta\Phi = P\delta w, \qquad \delta\Phi_{\Gamma 1} = \int_{\Gamma_p} \boldsymbol{p}\cdot\delta\boldsymbol{u}\, R\, d\varphi, \qquad \delta\Phi_{\Gamma 2} = \int_{\Gamma_c} \hat{\boldsymbol{p}}\cdot\delta\hat{\boldsymbol{u}}\, R_1\, d\theta.$$

Summation is performed over repeated subscripts i, $j = 1$ and 2 and integration is performed over the regions Ω_1 and Ω_2 occupied by the plate and the insert, respectively; $\boldsymbol{p}$ and $\hat{\boldsymbol{p}}$ are the vectors of the forces acting on Γ_p and Γ_c from the side opposite to the center of the insert; $\delta\boldsymbol{u}$ and $\delta\hat{\boldsymbol{u}}$ are the vectors of displacement variations (the vectors on Γ_c are denoted by a hat).

For the entire structure comprising the plate and the insert, we have

$$\delta\Phi = \delta\Phi_E + \delta\Phi_f, \qquad \delta\Phi_E = \delta\Phi_{E1} + \delta\Phi_{E2}, \tag{3.2}$$

i.e., $\delta\Phi$ is the sum of the stress work and the work of the frictional forces acting in the contact zone with displacement variations $\delta\Phi_f$ (taken with the minus sign). Without giving the expression for $\delta\Phi_f$, we substitute $\delta\Phi_{E1}$ and $\delta\Phi_{E2}$ from (3.1) into (3.2) to obtain the relation

$$\delta\Phi_f = \int\limits_{\Gamma_p} \boldsymbol{p}\cdot\delta\boldsymbol{u}\, R\, d\varphi - \int\limits_{\Gamma_c} \hat{\boldsymbol{p}}\cdot\delta\hat{\boldsymbol{u}}\, R_1\, d\theta, \tag{3.3}$$

which holds for any boundary conditions on the contours of the plate and the insert outside Γ_p and Γ_c. Below it is used to formulate the boundary conditions on Γ_p and Γ_c.

4. Interaction Between the Plate and the Insert. Equating the Cartesian coordinates of any points $\varphi \in \Gamma_p$ and $\hat{\theta} \in \Gamma_c$ that come in contact upon loading of the plate, we have

$$R\cos\varphi + u_1 = R_1\cos\hat{\theta} - c + \hat{u}_1, \qquad R\sin\varphi + u_2 = R_1\sin\hat{\theta} + \hat{u}_2, \tag{4.1}$$

where the displacement vectors $\boldsymbol{u} = (u_1, u_2)$ and $\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2)$ are determined at the points φ and $\hat{\theta}$, respectively, and the displacements and forces specified on Γ_c are denoted by a hat. We now formulate two versions of boundary conditions on Γ_p and Γ_c.

Version 1. We consider the point $\theta \in \Gamma_c$ which is closest to the point φ and for which

$$\cos\theta = \rho_s^{-1}(\cos\varphi + \varepsilon), \qquad \sin\theta = \rho_s^{-1}\sin\varphi, \qquad \rho_s = (1 + 2\varepsilon\cos\varphi + \varepsilon^2)^{1/2}.$$

We expand the right sides of (4.1) in Taylor series at the point θ. Neglecting products of derivatives of the displacements $\hat{u}_1$ and $\hat{u}_2$ by $(\hat{\theta} - \theta)$ and terms containing powers of $(\hat{\theta} - \theta)$ higher than the first power, we arrive at the approximate relations

$$\begin{aligned} R\cos\varphi + u_1 &= R_1[\cos\theta - (\hat{\theta} - \theta)\sin\theta] - c + \hat{u}_1, \\ R\sin\varphi + u_2 &= R_1[\sin\theta + (\hat{\theta} - \theta)\cos\theta] + \hat{u}_2. \end{aligned} \tag{4.2}$$

Here, in contrast to (4.1), the values of $\hat{u}_1$ and $\hat{u}_2$ are specified at the point θ. We find the projections of the displacement vectors $u_\rho = \boldsymbol{u}\cdot\boldsymbol{n}_c$, $u_\theta = \boldsymbol{u}\cdot\boldsymbol{l}_c$, $\hat{u}_\rho = \hat{\boldsymbol{u}}\cdot\boldsymbol{n}_c$, and $\hat{u}_\theta = \hat{\boldsymbol{u}}\cdot\boldsymbol{l}_c$ onto the normal $\boldsymbol{n}_c = (\cos\theta, \sin\theta)$ and the tangent $\boldsymbol{l}_c = (-\sin\theta, \cos\theta)$ to Γ_c at the point θ. From (4.2) we obtain the relations $u_\rho = u_{\rho s} + \hat{u}_\rho$ and $u_\theta = u_{\theta s} + \hat{u}_\theta$, in which the quantities $u_{\rho s} = R_1 - \rho_s R$ and $u_{\theta s} = R_1(\hat{\theta} - \theta)$ must be of the order of the displacements themselves.

The difference of the normal displacements $u_{\rho s}$ is equal to the distance between the points φ and θ taken with the minus sign. This difference is a known function of φ (or θ) that is independent of time. To eliminate penetration of Γ_p and Γ_c into one another outside the contact region during deformation, we require that the inequality $u_\rho - \hat{u}_\rho \geqslant u_{\rho s}$ hold for each pair of the points φ and θ. The values of $u_{\rho s}$ do not coincide with the values of the displacements $u_\rho = u_{\rho c} = -c\rho_s^{-1}(1 + \cos\varphi)$ specified in the case of an absolutely rigid insert [2]. We have $u_{\rho s} \leqslant u_{\rho c} \leqslant 0$, and the difference $(u_{\rho c} - u_{\rho s})$ is small (a quantity of order c^2). To ensure continuous transition to the contact conditions in [2] as the stiffness of the insert increases without bound, we make the nonpenetration condition stronger by setting $u_{\rho s} = u_{\rho c}$.

The difference of the tangential displacements $u_{\theta s}$ is equal, with one-place accuracy, to the length of the arc on Γ_c between the points θ and $\hat{\theta}$. The quantity $u_{\theta s}$ takes into account possible slippage of Γ_p and Γ_c about one another and changes in pairs of contacting points. Calculating $u_{\theta s}$, we approximately determine the coordinate of the point that is in contact with the point φ by the formula $\hat{\theta} = \theta + R_1^{-1}u_{\theta s}$. In the presence of friction, the quantity $u_{\theta s}$ depends at each time on the history of loading of the structure and it is treated, together with displacements, as the sought function of φ (or θ) and time τ. In the contact region, the partial derivative $\dot{u}_{\theta s}$ with respect to τ is the slip velocity. If $\dot{u}_{\theta s} = 0$, then $\dot{\theta} = \dot{\hat{\theta}}$, the pair of contacting points does not change, and attachment occurs.

We denote the contact and contact-free regions on Γ_p by Γ_{p1} and Γ_{p2}, respectively. The regions on

TABLE 1

Version	u	v	p	q	$\hat{u}$	$\hat{v}$	$\hat{p}$	$\hat{q}$	u_s	v_s	γ	Γ_1	Γ_2	Γ
1	u_ρ	u_θ	p_ρ	p_θ	$\hat{u}_\rho$	$\hat{u}_\theta$	$\hat{p}_\rho$	$\hat{p}_\theta$	$u_{\rho s}$	$u_{\theta s}$	γ_1	Γ_{p1}	Γ_{p2}	Γ_p
2	u_r	u_φ	p_r	p_φ	$\hat{u}_r$	$\hat{u}_\varphi$	$\hat{p}_r$	$\hat{p}_\varphi$	u_{rs}	$u_{\varphi s}$	γ_2	Γ_{c1}	Γ_{c2}	Γ_c

Γ_c formed by the points closest to the corresponding points on Γ_{p1} and Γ_{p2} are denoted by Γ_{c1} and Γ_{c2}, respectively ($\Gamma_p = \Gamma_{p1} \cup \Gamma_{p2}$ and $\Gamma_c = \Gamma_{c1} \cup \Gamma_{c2}$). We determine the projections of the force vectors $p_\rho = \boldsymbol{p} \cdot \boldsymbol{n}_c$ and $p_\theta = \boldsymbol{p} \cdot \boldsymbol{l}_c$ at the point φ and $\hat{p}_\rho = \hat{\boldsymbol{p}} \cdot \boldsymbol{n}_c$ and $\hat{p}_\theta = \hat{\boldsymbol{p}} \cdot \boldsymbol{l}_c$ at the point θ onto the normal and the tangent to Γ_c. The quantities $\hat{p}_\rho = \hat{\sigma}_{\rho\rho}$, $\hat{p}_\theta = \hat{\sigma}_{\rho\theta}$, and $\hat{\sigma}_{\theta\theta}$ are the stress components on the contour of the insert Γ_c in the polar coordinates (ρ, θ). Let, with variation in displacements, attachment occur everywhere in the contact region: $\delta u_\rho = \delta \hat{u}_\rho$ and $\delta u_\theta = \delta \hat{u}_\theta$. Then, the work due to friction is equal to zero and from (3.3), with allowance for the relations $R\, d\varphi = \gamma_1 R_1\, d\theta$ and $\gamma_1 = \rho_s^2 (1-\varepsilon)^{-1}(1+\varepsilon\cos\varphi)^{-1}$, we obtain

$$\delta\Phi_f = 0 = \int\limits_{\Gamma_{c1}} [(\gamma_1 p_\rho - \hat{p}_\rho)\delta u_\rho + (\gamma_1 p_\theta - \hat{p}_\theta)\delta u_\theta] R_1 d\theta + \int\limits_{\Gamma_{c2}} [\gamma_1(p_\rho \delta u_\rho + p_\theta \delta u_\theta) - (\hat{p}_\rho \delta\hat{u}_\rho + \hat{p}_\theta \delta\hat{u}_\theta)] R_1\, d\theta.$$

In these integrals we set the coefficients of arbitrary variations in displacements equal to zero. Bearing in mind that in the contact region, the normal forces on Γ_p and Γ_c are negative because of compression, we arrive at the boundary conditions

$$\begin{gathered} u_\rho - \hat{u}_\rho = u_{\rho s}, \quad u_\theta - \hat{u}_\theta = u_{\theta s}, \quad \hat{p}_\rho = \gamma_1 p_\rho, \quad \hat{p}_\theta = \gamma_1 p_\theta, \quad p_\rho < 0 \quad \text{on} \quad \Gamma_{p1}, \\ p_\rho = p_\theta = \hat{p}_\rho = \hat{p}_\theta = 0, \quad u_\rho - \hat{u}_\rho \geqslant u_{\rho s} \quad \text{on} \quad \Gamma_{p2}. \end{gathered} \tag{4.3}$$

Here the values of u_ρ, u_θ, p_ρ, and p_θ are determined at the point φ and the values of $\hat{u}_\rho$, $\hat{u}_\theta$, $\hat{p}_\rho$, and $\hat{p}_\theta$ are determined at the point θ; for brevity, the regions on Γ_c are not indicated; $u_{\rho s} = u_{\rho c} = -c\rho_s^{-1}(1+\cos\varphi)$. By virtue of (4.3), we have $\hat{\boldsymbol{p}} = \gamma_1 \boldsymbol{p}$ on Γ_{p1} and $\hat{\boldsymbol{p}} = \boldsymbol{p} = \boldsymbol{0}$ on Γ_{p2}. The coefficient γ_1 takes into account the difference in dimensions between the contact sites at the edges of the plate and the insert and the angle between these sites in the initial undeformed state.

Version 2. We now expand the left sides of equalities (4.1) in a Taylor series at the point $\hat{\varphi} \in \Gamma_p$ which is closest to $\hat{\theta}$ and for which

$$\cos\hat{\varphi} = r_s^{-1}[(1-\varepsilon)\cos\hat{\theta} - \varepsilon], \qquad \sin\hat{\varphi} = r_s^{-1}(1-\varepsilon)\sin\hat{\theta},$$
$$r_s = [1 - 2\varepsilon(1-\varepsilon)(1+\cos\hat{\theta})]^{1/2}.$$

Further, the boundary conditions are formulated similarly to version 1 but for the following projections of the displacement and force vectors onto the normal $\boldsymbol{n}_p = (\cos\hat{\varphi}, \sin\hat{\varphi})$ and the tangent $\boldsymbol{l}_p = (-\sin\hat{\varphi}, \cos\hat{\varphi})$ to Γ_p: $u_r = \boldsymbol{u} \cdot \boldsymbol{n}_p$, $u_\varphi = \boldsymbol{u} \cdot \boldsymbol{l}_p$, $p_r = \boldsymbol{p} \cdot \boldsymbol{n}_p$, and $p_\varphi = \boldsymbol{p} \cdot \boldsymbol{l}_p$ at the point $\hat{\varphi}$ and $\hat{u}_r = \hat{\boldsymbol{u}} \cdot \boldsymbol{n}_p$, $\hat{u}_\varphi = \hat{\boldsymbol{u}} \cdot \boldsymbol{l}_p$, $\hat{p}_r = \hat{\boldsymbol{p}} \cdot \boldsymbol{n}_p$, and $\hat{p}_\varphi = \hat{\boldsymbol{p}} \cdot \boldsymbol{l}_p$ at the point $\hat{\theta}$. The quantities $p_r = \sigma_{rr}$, $p_\varphi = \sigma_{r\varphi}$, and $\sigma_{\varphi\varphi}$ are the stress components on Γ_p in the plate in the polar coordinates (r, φ). We obtain the equalities $u_r = u_{rs} + \hat{u}_r$ and $u_\varphi = u_{\varphi s} + \hat{u}_\varphi$, in which $u_{rs} = R(r_s - 1)$ and $u_{\varphi s} = R(\hat{\varphi} - \varphi)$. To ensure continuous transition to the contact conditions of [2] as the stiffness of the insert increases without bound, we make the nonpenetration condition $u_r - \hat{u}_r \geqslant u_{rs}$ stronger by setting $u_{rs} = u_{rc} = -c(1+\cos\hat{\varphi})$. We determine the contact region Γ_{c1} and the contact-free region Γ_{c2} on Γ_c and the regions Γ_{p1} and Γ_{p2} on Γ_p formed by the points closest to the corresponding points from Γ_{c1} and Γ_{c2}. In this case, the regions Γ_{p1}, Γ_{p2}, Γ_{c1}, and Γ_{c2} can differ from those in version 1. We have $\Gamma_p = \Gamma_{p1} \cup \Gamma_{p2}$ and $\Gamma_c = \Gamma_{c1} \cup \Gamma_{c2}$. The force vectors in the contact region are related by the formula $\hat{\boldsymbol{p}} = \gamma_2 \boldsymbol{p}$, where $\gamma_2 = r_s^{-2}[1 - \varepsilon(1+\cos\hat{\theta})]$.

We use the notation given in the first row of Table 1, which is common for both versions, for the quantities and regions listed in the last two rows of Table 1. Using the common notation, we write the boundary conditions for versions 1 and 2 in the form

$$
\begin{gathered}
u-\hat{u}=u_s, \quad v-\hat{v}=v_s, \quad \hat{p}=\gamma p, \quad \hat{q}=\gamma q, \quad p<0 \quad \text{on} \quad \Gamma_1, \\
p=q=\hat{p}=\hat{q}=0, \quad u-\hat{u}\geqslant u_s \quad \text{on} \quad \Gamma_2.
\end{gathered}
\tag{4.4}
$$

In version 1, these conditions coincide with (4.3). For the contact region, from $p<0$ it follows that $\hat{p}<0$ and the nonpenetration condition $u-\hat{u}\geqslant u_s$ holds everywhere on $\Gamma=\Gamma_1\cup\Gamma_2$. Boundary conditions (4.4) are formulated irrespective of the materials of the plate and insert, the ratio of their stiffnesses, and the properties of the contacting surfaces. Below, they are supplemented by boundary conditions that take friction into account.

We now use relation (3.3) not for displacement variations but for their velocities. Replacing $\delta\Phi_f$ by $\dot{\Phi}_f$ and integrating over Γ_c, we obtain

$$
\dot{\Phi}_f=\int_{\Gamma_c}[\gamma(p\dot{u}+q\dot{v})-(\hat{p}\dot{\hat{u}}+\hat{q}\dot{\hat{v}})]\,d\Gamma_c.
$$

Next, using the equalities $\hat{p}=\gamma p$, $\hat{q}=\gamma q$, $\dot{u}=\dot{\hat{u}}$, and $\dot{v}-\dot{\hat{v}}=\dot{v}_s$, which follow from (4.4) for the contact region, we obtain the following relation for energy power dissipated in friction:

$$
\dot{\Phi}_f=\int_{\Gamma_{p1}} q\dot{v}_s\,d\Gamma_p=\int_{\Gamma_{c1}}\hat{q}\dot{v}_s\,d\Gamma_c.
$$

The energy power densities per unit length of the contours of the plate ($Q=q\dot{v}_s$) and the insert ($\hat{Q}=\hat{q}\dot{v}_s$) and the quantity $\dot{\Phi}_f$ must be nonnegative.

5. Contact Problems with Allowance for Coulomb Friction. Let Coulomb friction [3] act in the contact region Γ_1. The forces on Γ_1 should then satisfy the inequalities $p<0$ and $|q|\leqslant\mu|p|$ or $f_1=\mu p+q\leqslant 0$, $f_2=\mu p-q\leqslant 0$, and $F=f_1f_2\geqslant 0$, where μ is the friction coefficient. In Cartesian coordinates, the region occupied by p and q in the half-plane $p<0$ is bounded by the straight lines $f_1=0$ and $f_2=0$, on which the vector (p,q) is inclined to Γ at minimum possible angles. The friction law does not limit the magnitude of the vector (p,q). The restrictions imposed on $\hat{p}$ and $\hat{q}$ by the friction law are satisfied if the relations $\hat{p}=\gamma p$ and $\hat{q}=\gamma q$ hold. Therefore, they are not included in the boundary conditions below.

The plate edge can slide over the insert with friction at nonzero velocity $\dot{v}_s$ only if the values of the forces are on one of the boundary lines $f_1=0$ or $f_2=0$, the energy dissipation power density is nonnegative $Q=q\dot{v}_s\geqslant 0$, and the frictional force acts on each of the contacting bodies in the direction opposite to the velocity of slippage of this body over the other. The magnitudes of the slip velocity $\dot{v}_s$ can be arbitrary and independent of the forces p and q. At the remaining points in the contact region, at which the slip conditions do not hold, there is attachment $\dot{v}_s=0$.

We arrive at the boundary conditions

$$
\begin{gathered}
u-\hat{u}=u_s, \quad \dot{v}_s=0, \quad \hat{p}=\gamma p, \quad \hat{q}=\gamma q, \quad p<0, \quad F>0 \quad \text{on} \quad \Gamma_1', \\
u-\hat{u}=u_s, \quad f=0, \quad \hat{p}=\gamma p, \quad \hat{q}=\gamma q, \quad p<0, \quad Q\geqslant 0 \quad \text{on} \quad \Gamma_1'', \\
p=q=\hat{p}=\hat{q}=0, \quad u-\hat{u}\geqslant u_s \quad \text{on} \quad \Gamma_2.
\end{gathered}
\tag{5.1}
$$

Here, in view of the rigorous condition $p<0$, the condition $F=f_1f_2>0$ on Γ_1' is equivalent to the two inequalities $f_1<0$ and $f_2<0$, which are linear in the sought functions. Attachment occurs on Γ_1' and at those points on Γ_1'' at which $Q=0$. At the remaining points on Γ_1'', we have slip $Q>0$. As a function f at each point on Γ_1'', we use the function f_1 or f_2, which are equal to zero at this point. For $f=f_1=0$ and $q=-\mu p>0$, the inequality $Q\geqslant 0$ can be replaced by $\dot{v}_s\geqslant 0$, and for $f=f_2=0$ and $q=\mu p<0$, it can be replaced by $\dot{v}_s\leqslant 0$.

The partition $\Gamma=\Gamma_1\cup\Gamma_2$ and $\Gamma_1=\Gamma_1'\cup\Gamma_1''$ is completely determined by the values of the forces p and q at each current time. The regions Γ_1', Γ_1'', and Γ_2, the form of the function f (f_1 or f_2) on Γ_1'', and the values of the forces p and q on Γ at each time depend on the history of loading of the structure and attachment and slip of the contacting surfaces of the plate and the insert and are found from the solution of the problem. The conditions for the velocities of tangential displacements in (5.1) are used to trace the loading history.

In contrast to [2], the condition $\dot{f} < 0$ on Γ_1'', which implies discontinuous change in f as a function of time, is ignored in (5.1) and hence, one need not specify $\dot{v}_s = 0$. In the numerical solutions given below, as in [2], the case $\dot{f} < 0$ on Γ_1'' is not realized.

It should be noted that boundary conditions (5.1) are satisfied not at the actually contacting points, but at pairs of the points φ and θ for version 1 or $\hat{\varphi}$ and $\hat{\theta}$ for version 2 that are specified according to the proposed approximate formulation of the problem. Therefore, during solution for both versions, it suffices to determine the attachment, slip, contact, and contact-free regions on just one contour Γ.

Thus, we have two contact problems for Eqs. (1.1) with boundary conditions (2.1), (2.2), and (5.1) in versions 1 and 2. The solutions of these problems at each time depend on the loading history and the interaction between the plate and the insert.

Setting $\varepsilon = 0$ in (5.1), we arrive at the contact problem with friction for zero clearance. If w increases monotonically (from zero in the initial undeformed state of the structure), the solution of the problem varies linearly in w and can be found for the boundary conditions

$$\begin{gathered} u = \hat{u}, \quad v = \hat{v}, \quad p = \hat{p}, \quad q = \hat{q}, \quad p < 0, \quad F > 0 \quad \text{on} \quad \Gamma_1', \\ u = \hat{u}, \quad f = 0, \quad p = \hat{p}, \quad q = \hat{q}, \quad p < 0, \quad Q_1 \geqslant 0 \quad \text{on} \quad \Gamma_1'', \\ p = q = \hat{p} = \hat{q} = 0, \quad u \geqslant \hat{u} \quad \text{on} \quad \Gamma_2. \end{gathered} \tag{5.2}$$

Here the values of u, v, p, and q for the plate and $\hat{u}$, $\hat{v}$, $\hat{p}$, and $\hat{q}$ for the insert are determined at the same point on $\Gamma = \Gamma_p = \Gamma_c$ in the projections onto the normal and the tangent to Γ at the same point. The regions Γ_1', Γ_1'', and Γ_2 form at the initial time for any arbitrarily small w and remain unchanged as w increases. In the region Γ_1', by virtue of $\dot{v}_s = 0$, we have $v_s = 0$ and $v = \hat{v}$. At each point of Γ_1'', the function f remains the same (f_1 or f_2) for any $w > 0$. With allowance for the linear dependence of q and v_s on w, the inequality $Q \geqslant 0$ is replaced by the condition of nonnegative density of the energy dissipated in friction, i.e., $Q_1 = 0.5qv_s \geqslant 0$.

In the absence of friction $\mu = 0$, conditions (5.1) lead to the boundary conditions

$$\begin{gathered} u - \hat{u} = u_s, \quad \hat{p} = \gamma p, \quad q = \hat{q} = 0, \quad p < 0 \quad \text{on} \quad \Gamma_1, \\ p = q = \hat{p} = \hat{q} = 0, \quad u - \hat{u} \geqslant u_s \quad \text{on} \quad \Gamma_2, \end{gathered} \tag{5.3}$$

determined in versions 1 and 2. The problem for Eqs. (1.1) with boundary conditions (2.1), (2.2), and (5.3) has a unique solution for each version. The strain energy for the plate and the insert Φ_E, which is treated as a functional, reaches a minimum value for this solution in the displacement space satisfying the boundary conditions for displacements in (2.1) and (2.2) and the nonpenetration condition $u - \hat{u} \geqslant u_s$ everywhere on Γ. The regions Γ_1 and Γ_2 are determined from the solution of the problem.

The problems for plates with an absolutely rigid insert have been solved ignoring friction [4-6] and with allowance for friction [2]. Hyer and Klang [7] obtained a series solution for an infinite plate with a deformable insert in the presence of a clearance and friction with the difference of displacements of the hole edge and the insert contour specified in the attachment region ignoring the history of interaction between the plate and the insert.

6. Algorithm of Solution of Contact Problems. Solutions are found for monotonically increasing displacement w of the right side of the plate using the following algorithm for any version of the boundary conditions considered.

On Γ, we introduce a new variable — the coordinate η. It has the form $\eta = 1 - \varphi/\pi$ in version 1 and $\eta = 1 - \hat{\theta}/\pi$ in version 2 and increases along Γ in the clockwise direction ($0 \leqslant \eta \leqslant 1$) (Fig. 1). We assume that, at any time, the regions Γ_1', Γ_1'', and Γ_2 on Γ occupy the segments $0 \leqslant \eta < b$, $b \leqslant \eta < l$, and $l \leqslant \eta \leqslant 1$, respectively, and the contact region $\Gamma_1 = \Gamma_1' \cup \Gamma_1''$ occupies the segment $0 \leqslant \eta < l$. At all points on Γ_1'', the same function f (f_1 or f_2) is assumed to be zero and, hence, the frictional forces are assumed to act along Γ_1'' in the same direction. The values of b and l and the form of the function f (f_1 or f_2), which is equal to zero on Γ_1'', depend on the displacement w, the clearance c, the friction coefficient μ, the ratio of the stiffnesses of the plate and the insert, and the version 1 or 2 of boundary conditions.

In the calculations, the length l of the contact region is increased by steps during loading. In each step from τ to $\tau + \Delta\tau$, the value of l is assigned at the end of the step at time $\tau + \Delta\tau$. The slip velocities at the end of the step are determined from the formula $\dot{v}_s = (v_s - v_{s\tau})/\Delta\tau$ (the values at the beginning of the step are denoted by the subscript τ; at the initial time, $l = 0$ and $v_s = 0$ everywhere on Γ).

Discarding the inequalities in (5.1) and satisfying the condition $\dot{v}_s = 0$ on Γ_1', we arrive at the boundary conditions

$$
\begin{gathered}
u - \hat{u} = u_s, \quad v - \hat{v} = v_{s\tau}, \quad \hat{p} = \gamma p, \quad \hat{q} = \gamma q \quad \text{for} \quad 0 \leqslant \eta < b, \\
u - \hat{u} = u_s, \quad f = 0, \quad \hat{p} = \gamma p, \quad \hat{q} = \gamma q \quad \text{for} \quad b \leqslant \eta < l, \\
p = q = \hat{p} = \hat{q} = 0 \quad \text{for} \quad l \leqslant \eta \leqslant 1.
\end{gathered}
\tag{6.1}
$$

Here the differences of tangential displacements $v_{s\tau}$ found in the previous time step are specified on the segment $0 \leqslant \eta < b$. Solution of the problem for Eqs. (1.1) subject to boundary conditions (2.1), (2.2), and (6.1) (referred to as problem A) gives the state of equilibrium of the structure at the end of the step with a specified contact region.

We introduce two auxiliary problems A1 and A2, which differ from problem A in that the boundary conditions specify

$$
\begin{gathered}
u = \hat{u}, \quad v = \hat{v} \text{ in A1}, \quad u - \hat{u} = c^{-1}u_s, \quad v - \hat{v} = c^{-1}v_{s\tau} \text{ in A2} \quad \text{for} \quad 0 \leqslant \eta < b, \\
u = \hat{u}, \quad f = 0 \text{ in A1}, \quad u - \hat{u} = c^{-1}u_s, \quad f = 0 \text{ in A2} \quad \text{for} \quad b \leqslant \eta < l, \\
u_1 = 1, \quad u_2 = 0 \text{ in } A1, \quad u_1 = u_2 = 0 \text{ in } A2 \quad \text{for} \quad x_1 = L_2, \quad 0 \leqslant x_2 \leqslant H.
\end{gathered}
$$

The solution of the contact problem without a clearance for Eqs. (1.1) and boundary conditions (2.1), (2.2), and (5.2) is determined as the solution of problem A1 for $\varepsilon = 0$, $w = 1$, and values of b, l, and f such that the inequalities in (5.2) hold.

The plate and the insert are divided into Lagrangian finite elements (quadrilateral, nine-node, and isoparametric) [8], as shown (for $\varepsilon = 0$) in Fig. 1. The sets of nodes on Γ_p and Γ_c are composed of pairs of points (φ and θ in version 1 or $\hat{\varphi}$ and $\hat{\theta}$ in version 2) at which the values of the sought functions are related by the specified boundary conditions. In other respects, the algorithm is similar to the one given in [2] for problems with an absolutely rigid insert. The systems of finite-element equations for problems A1 and A2 are formulated using the virtual displacement principle. These systems have the same unsymmetrical matrix of coefficients of the unknown variables (components of nodal displacements) and are solved by the Gauss method of elimination [8, 9] with allowance for the band nature of this matrix and the fact that most of its coefficients are symmetric about the principal diagonal. We seek a solution of problem A as a sum of solutions of problems A1 and A2 with the coefficients w and c. Using the Boussinesq principle [3, 10], we determine w from the condition of zero normal force p at the extreme right node in the contact region $\eta = l_*$. Iterations are used to determine the location of the extreme right node in the attachment region $\eta = b_*$ at which the force function f, which is equal to zero in the slip region, vanishes at this node (the values b_* and l_* are used as the lengths of the attachment and contact regions $b = b_*$ and $l = l_*$). As a result, the desired solution of the contact problem is determined, and inequalities (5.1) hold. The conditions $v - \hat{v} = v_{s\tau}$ and $f = 0$ are satisfied at the node $\eta = b_*$, and the conditions $f = 0$ and $p = 0$ are satisfied at the node $\eta = l_*$. In this sense, there is continuous transition along Γ from one set of boundary conditions to the other.

7. Calculation Results. We convert to dimensionless quantities. For this, we multiply x_1, x_2, r, and ρ by the normalizing factor R^{-1}, the displacements and the clearance c by L_0^{-1}, the strains by $\omega = RL_0^{-1}$, the stresses and the forces p and q by ωE_1^{-1}, and the strain energies of the plate Φ_{E1} and the insert Φ_{E2}, the energy Φ_f dissipated in friction, and the work Φ of the tensile force by $E_1^{-1}L_0^{-2}$ (L_0 is a constant having the dimension of length). The previous notation is used for the dimensionless quantities. In this case, $R = 1$, $R_1 = 1 - \varepsilon$, $R_2 = 0.25$, $H = L_1 = 2.5$, $L_2 = 5$, and $c = \omega\varepsilon$. The Poisson's ratio is $\nu_1 = \nu_2 = 0.3$.

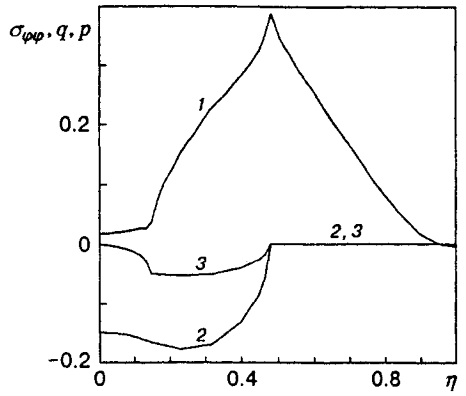

Fig. 2

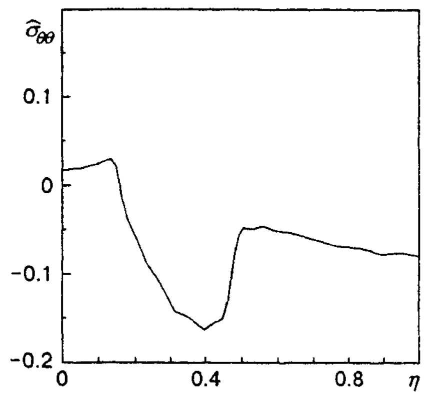

Fig. 3

Fig. 2. Hoop stress $\sigma_{\varphi\varphi}$ (curve 1) and normal p and tangential q forces (curves 2 and 3) on the hole contour in the plate for the solution of the problem given in Fig. 1.

Fig. 3. Hoop stress $\hat{\sigma}_{\theta\theta}$ on the insert contour Γ_c for the solution of the problem given in Figs. 1 and 2.

The dimensionless equations and the boundary conditions include the two parameters c and ε, related to the magnitude of the clearance. From specified c and ε, we determine $\omega = c\varepsilon^{-1}$ and convert from the dimensionless sought functions to dimensional quantities. In the absence of a clearance ($c = \varepsilon = 0$), the solution is linearly proportional to w, and, hence, we set $w = 1$. Reverting to the dimensional quantities and specifying the dimensional parameters w and R, we have $L_0 = w$ and $\omega = RL_0^{-1}$. In a similar manner, we can convert from the dimensionless sought functions to dimensional functions.

For the contact problem with friction in the absence of a clearance for Eqs. (1.1) and boundary conditions (2.1), (2.2), and (5.2), we examine a solution that was obtained for $\varepsilon = 0$, $\mu = 0.3$, $E_1 = E_2$, $w = 1$, $l = 0.4807$, and $b = 0.1485$ ($\pi l = 86.5°$ and $\pi b = 26.7°$). The finite-element grids for the plate and the insert (Lagrangian isoparametric nine-node finite elements are used) in the initial and deformed states are shown in Fig. 1. The coordinates of the nodes X_i in the deformed state are related to their initial coordinates x_i by the formula $X_i = x_i + \beta u_i$ ($i = 1$ and 2), where the coefficient β is chosen so that, the product of β by the maximum magnitude of the components of the global vector of the sought unknowns — the nodal displacements — is equal to unity. Conversion to the dimensionless displacements and multiplication of them by β result in overestimated strains of the plate and the insert (Fig. 1).

The tensile force $P = 0.195$ and the strain energies of the plate and the insert $\Phi_{E1} = 0.07504$ and $\Phi_{E2} = 0.01996$ exceed the ones obtained in the solution in the absence of friction, in which $P = 0.184$, $\Phi_{E1} = 0.07322$, $\Phi_{E2} = 0.01874$, $l = 0.4708$, and $b = 0$. As the stiffness of the insert increases for $\mu = 0.3$, the energy dissipated in friction increases from $\Phi_f = 0.0025$ in the solution considered to $\Phi_f = 0.0053$ in the case of an absolutely rigid insert with $\Phi_{E1} = 0.1183$.

Figure 2 shows p, q, and $\sigma_{\varphi\varphi}$ in the plate and Fig. 3 shows $\hat{\sigma}_{\theta\theta}$ in the insert on Γ as functions of η. We have $p = q = 0$ for $l \leqslant \eta \leqslant 1$. The maximum value $\sigma_{\varphi\varphi} = \sigma^*_{\varphi\varphi}$ is reached on the free part of Γ in the vicinity of the point $\eta = l$, and the maximum absolute value of p is reached at the interior point of the contact region. In contrast to the case of an absolutely rigid insert [2], we have $\sigma_{\varphi\varphi} > 0$ in the neighborhood of the point $\eta = 0$. Because of friction, the values of $\sigma^*_{\varphi\varphi}$ and the stress-concentration coefficient $K = HP^{-1}\sigma^*_{\varphi\varphi}$ increase from 0.3282 and 4.460, respectively, for $\mu = 0$ to 0.3873 and 4.965, respectively, for $\mu = 0.3$.

As E_2/E_1 decreases for $\varepsilon = 0$, $\mu = 0.3$, and $w = 1$, the values of P, Φ_E, and (Φ_f/Φ) and the maxima $\sigma_{\varphi\varphi}$ and $\hat{\sigma}_{\theta\theta}$ on Γ decrease, whereas the values of l and b increase. We have $l = 0.4748$ and $b = 0.02366$ for $E_2 = 16E_1$ and $l = 0.4868$ and $b = 0.2054$ for $E_2 = 0.25E_1$.

A solution of the contact problem in the presence of a clearance and friction for Eqs. (1.1) with

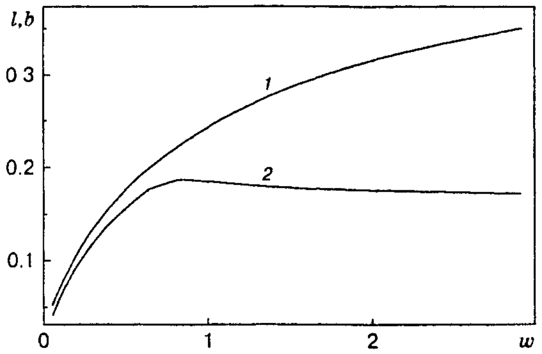

Fig. 4

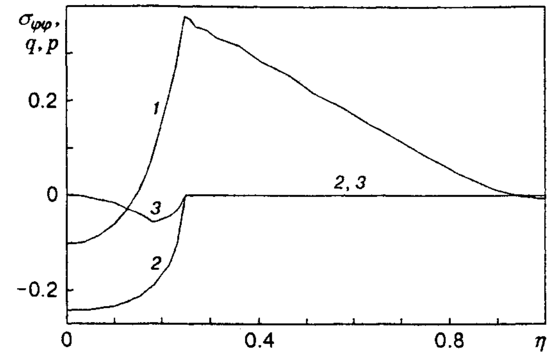

Fig. 5

Fig. 4. Lengths of the contact region l (curve 1) and the attachment region b (curve 2) versus the displacement w of the right side of the plate in the problem with boundary conditions in version 2 in the presence of a clearance and friction ($\varepsilon = 0.05$, $\mu = 0.3$, and $E_1 = E_2$).

Fig. 5. Hoop stress $\sigma_{\varphi\varphi}$ (curve 1) and normal p and tangential q forces (curves 2 and 3) on the hole contour in the plate in the solution of the problem given in Fig. 4 for $l = 0.25$, $b = 0.1824$, and $w = 1.1062$.

TABLE 2

Solution	μ	l	b	$P \cdot 10^2$	$\Phi_{E1} \cdot 10^3$	$\Phi_{E2} \cdot 10^3$	$\sigma^*_{\varphi\varphi}$	K
$U^{(1)}$	0.3	0.1455	0.1309	4.651	6.411	1.833	0.1017	5.468
$U^{(2)}$	0.3	0.15	0.1339	4.694	6.481	1.836	0.1029	5.483
$U^{(3)}$	0	0.1499	0	4.679	6.427	1.874	0.0994	5.313

boundary conditions (2.1), (2.2), and (5.1) in version 1 was obtained for $c = 1$, $\varepsilon = 0.05$, $\mu = 0.3$, and $E_1 = E_2$ in 16 steps variable in l ($l = 0.05$–0.45). For $l = 0.05$, zero differences of tangential displacements $v_s = 0$ were assigned in the attachment region. Curves 1 and 2 in Fig. 4 show the parameters l and b as functions of w. In contrast to the case of an absolutely rigid insert [2], the signs of q and $\dot{v}_s$ in the slip region do not change during loading and the plate edge slides over the insert contour in the clockwise direction. As w increases, the length of the attachment region reaches a maximum $b = 0.1869$ ($l = 0.225$) and $w = 0.8445$ and then decreases (curve 2 in Fig. 4) but not as rapidly as in the solution for an absolutely rigid insert [2]. The dissipated energy Φ_f is negligibly small for $l \leqslant 0.225$, and then it sharply increases but remains well below the strain energies of the plate and the insert. For $l = 0.45$, it is approximately 1.6% of $\Phi_E = \Phi_{E1} + \Phi_{E2}$.

Figure 5 shows p, q, and $\sigma_{\varphi\varphi}$ on Γ_p for $l = 0.25$, $b = 0.1824$, $w = 1.1062$, and $P = 0.1559$. It can be seen that $\sigma_{rr} < 0$ and $\sigma_{\varphi\varphi} < 0$ in the neighborhood of the point $\eta = 0$.

For the same value of w, the difference between the solutions of the problems for Eqs. (1.1) and boundary conditions (2.1), (2.2), and (5.1) of versions 1 and 2 (the corresponding solution vectors are denoted by $\boldsymbol{U}^{(1)}$ and $\boldsymbol{U}^{(2)}$, respectively) is small. For small values of w, these solutions differ insignificantly from solutions ignoring friction. Table 2 lists the values of μ, l, b, P, Φ_{E1}, and Φ_{E2}, the maximum hoop stress at the edge of the hole $\sigma^*_{\varphi\varphi}$, and the stress-concentration coefficient $K = HP^{-1}\sigma^*_{\varphi\varphi}$ calculated for $c = 1$, $\varepsilon = 0.05$, $E_1 = E_2$, and $w = 0.3838$ in the solutions $\boldsymbol{U}^{(1)}$ and $\boldsymbol{U}^{(2)}$ for $\mu = 0.3$ and in the solution $\boldsymbol{U}^{(3)}$ of the problem for Eqs. (1.1) with boundary conditions (2.1), (2.2), and (5.3) in version 2 for $\mu = 0$. The boundary conditions in version 1 are less restrictive for the plate and the insert than the boundary conditions in version 2: the values of l, b, P, Φ_{E1}, Φ_{E2}, $\sigma^*_{\varphi\varphi}$, and K in $\boldsymbol{U}^{(1)}$ are lower than those in $\boldsymbol{U}^{(2)}$.

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